

Existence and multiplicity results for fractional p -Kirchhoff equation with sign changing nonlinearities

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Abstract

In this paper, we show the existence and multiplicity of nontrivial, non-negative solutions of the fractional p -Kirchhoff problem

$$M \left(\int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} dx dy \right) (-\Delta)_p^s u = \lambda f(x) |u|^{q-2} u + g(x) |u|^{r-2} u \text{ in } \Omega,$$

$$u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

where $(-\Delta)_p^s$ is the fractional p -Laplace operator, Ω is a bounded domain in \mathbb{R}^n with smooth boundary, $f \in L^{\frac{r}{r-q}}(\Omega)$ and $g \in L^\infty(\Omega)$ are sign changing, M is continuous function, $ps < n < 2ps$ and $1 < q < p < r \leq p_s^* = \frac{np}{n-ps}$.

Key words: Fractional p -Laplacian, Kirchhoff type problem, critical exponent problem

1 Introduction

In this work, we study the existence and multiplicity of solutions for the following p -Kirchhoff equation

$$(P_\lambda) \quad \begin{cases} M \left(\int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} dx dy \right) (-\Delta)_p^s u = \lambda f(x) |u|^{q-2} u + g(x) |u|^{r-2} u, & u \geq 0, u \not\equiv 0 \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $(-\Delta)_p^s$ is the fractional p -Laplace operator defined as

$$(-\Delta)_p^s u(x) = -2 \int_{\mathbb{R}^n} \frac{|u(y) - u(x)|^{p-2} (u(y) - u(x))}{|x - y|^{n+ps}} dy$$

where $M(t) = a + bt$, $a, b > 0$, $p \geq 2$, $1 < q < p < r \leq p_s^*$, $ps < n < 2ps$ with $s \in (0, 1)$, λ is a positive parameter, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and f and g satisfy the

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following assumptions,

(f1) $f \in L^{\frac{r}{r-q}}(\Omega)$ and $g \in L^\infty(\Omega)$ and **(f2)** $f^+(x) = \max\{f(x), 0\} \not\equiv 0$ in $\bar{\Omega}$ and $g^+(x) = \max\{g(x), 0\} \not\equiv 0$ in $\bar{\Omega}$ (f and g are possibly sign changing on $\bar{\Omega}$). Fractional p -Laplacian operator is quasilinear generalization of fractional Laplace operator $(-\Delta)_2^s$.

In this work, we would like to study the existence and multiplicity of solutions for the problem (P_λ) with sign changing convex-concave type nonlinearity. A subcritical problem of this type is studied in [17] with $M = 1$. The presence of $M(\|u\|_{X_0}^p)$ in the weak formulation requires the compactness of Palais-Smale sequences to obtain the solution. In order to show the existence of solution, we need to show the strong convergence of weak limit of Palais-Smale sequence which is not the case for $M \equiv 1$. We prove the strong convergence of Palais-Smale sequences using the concentration compactness [22]. The growth of Kirchhoff term makes the study of fibering analysis more complicated. Starting from the pioneering work of Ambrosetti-Brezis-Cerami [3], there are many works on multiplicity results with convex and concave type nonlinearities. These results were generalized to p -Laplacian type equations by many authors see [9], [16] and references therein.

During the last one decade, several authors used the Nehari manifold and associated fiber maps to solve semilinear and quasilinear elliptic problems when the nonlinearity changes sign. In [6, 28], authors studied the existence and multiplicity of solutions for subcritical and critical nonlinearities via Nehari manifold method for semilinear equation and in [12], these results are studied for p -Laplacian equation. In [11], authors considered the following Kirchhoff equation:

$$M\left(\int_{\Omega} |\nabla u|^2 dx\right) \Delta u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p-2}u, \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $1 < q < 2 < p < 2^*$ and $f(x)$ and $g(x)$ are sign changing continuous functions. They studied the existence and multiplicity results by studying the structure of Nehari manifold in different cases (i) $p < 4$ (ii) $p = 4$ and (iii) $p > 4$. The above problem is called non-local because of the presence of integral over Ω . It means that (P_λ) is no more a point wise identity. Non-local operators, naturally arise in several physical and biological models such as continuum mechanics, phase transition phenomena, population dynamics and game theory (see [7] and references therein).

The fractional Laplacian operator has been a classical topic in Fourier analysis and nonlinear partial differential equations for a long time. Fractional operators are also involved in financial mathematics, where Levy processes with jumps appear in modeling the asset prices (see [4]). The critical exponent problems for square root of Laplacian are studied in [10], [26]. The Brezis-Nirenberg type results for fractional Laplacian $(-\Delta)^s$, ($0 < s < 1$) are studied in [21], [24], [25]. In [29], authors studied the Nehari manifold for square root of Laplacian by considering the harmonic extension of solutions in the half cylinder $\Omega \times (0, \infty)$ vanishing on the boundary $\partial\Omega \times [0, \infty)$. The idea of these harmonic extensions was initially introduced and studied in the beautiful work of Caffarelli and Silvestre[8]. In [17], existence of solutions for p -fractional Laplacian equations with sign changing nonlinearities are studied via the method of Nehari manifold. The Kirchhoff

equations with Laplacian and p -Laplacian operators are widely studied by many authors using variational methods (see [1], [2], [14], [18] and references therein).

In [15], authors considered the following Kirchhoff equation with fractional Laplacian:

$$M \left(\int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right) (-\Delta)_2^s u = \lambda f(x, u) + |u|^{2_s^*-2} u \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

where λ is positive parameter and $f(x, t)$ is a subcritical term having superlinear growth at $t = 0$. They studied the existence of mountain pass solutions using the concentration compactness principle of Lions for fractional Sobolev spaces [22]. To the best of our knowledge, there are no works in the literature dealing with Nehari manifold and p -fractional equations with sign changing nonlinearities to study the existence and multiplicity results.

In this work, we prove the existence of multiple non-negative solutions for the Kirchhoff elliptic equation with p -fractional operator and sign changing nonlinearity by studying the nature of Nehari manifold with respect to the parameter λ and fiber maps associated with the Euler functional. In the subcritical case, we show the existence of local minimizers of the associated functional on nonempty decompositions of Nehari manifold. We show the multiplicity result by extracting Palais-Smale sequences in the Nehari manifold. The results obtained here are somehow expected but we show how the results arise out of nature of Nehari manifold. For this we adopt the approach in [11]. We also prove an existence result for critical case using concentration compactness Lemma for suitable range of λ .

The natural space to look for solutions is the fractional Sobolev space $W_0^{s,p}(\Omega)$. In order to study (P_λ) , it is important to encode the ‘boundary condition’ $u = 0$ in $\mathbb{R}^n \setminus \Omega$ in the weak formulation. It was observed in [23, 24] that the interaction between Ω and $\mathbb{R}^n \setminus \Omega$, which gives positive contribution in the norm $\|u\|_{W^{s,p}(\mathbb{R}^n)}$. They introduced new function spaces to study the variational functionals related to fractional Laplacian. Subsequently, in [17], authors studied the p -fractional equations on the function spaces defined as:

For $1 < p < \infty$, $p \neq 2$, define the space X as

$$X = \left\{ u \mid u : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is measurable, } u|_{\Omega} \in L^p(\Omega), \left(\frac{u(x) - u(y)}{|x - y|^{\frac{n}{p} + s}} \right) \in L^p(Q) \right\},$$

where $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega = \mathbb{R}^n \setminus \Omega$. Then the space X endowed with the norm, defined as

$$\|u\|_X = \left(\|u\|_{L^p(\Omega)} + \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} dx dy \right)^{1/p} \quad (1.1)$$

is a reflexive Banach space. It is immediate to observe that $C_c^1(\Omega) \subseteq X$. The function space X_0 denotes the closure of $C_c^\infty(\Omega)$ in X . The space X_0 is a Banach space which can be endowed with the norm, defined as

$$\|u\|_{X_0} = \left(\int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} dx dy \right)^{1/p}. \quad (1.2)$$

Note that in equations (1.1) and (1.2), the integrals can be extended to \mathbb{R}^{2n} , since $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$. The Energy functional associated to the problem (P_λ) is given by

$$\mathcal{J}_\lambda(u) = \frac{1}{p} \widehat{M}(\|u\|_{X_0}^p) - \frac{\lambda}{q} \int_{\Omega} f(x)|u|^q dx - \frac{1}{r} \int_{\Omega} g(x)|u|^r dx \quad (1.3)$$

where $\widehat{M}(t) = \int_0^t M(s)$ is the primitive of M .

Definition 1.1. A function $u \in X_0$ is called weak solution of (P_λ) if u satisfies

$$\begin{aligned} M(\|u\|_{X_0}^p) \int_{\mathbb{R}^{2n}} |u(x) - u(y)|^{p-2} (u(x) - u(y))(\varphi(x) - \varphi(y))|x - y|^{-n-ps} dx dy \\ = \lambda \int_{\Omega} f(x)|u|^{q-2} u \varphi(x) dx + \int_{\Omega} g(x)|u|^{r-2} u \varphi(x) dx \quad \text{for all } \varphi \in X_0. \end{aligned} \quad (1.4)$$

The Nehari set \mathcal{N}_λ associated to the problem (P_λ) is defined as

$$\mathcal{N}_\lambda = \{u \in X_0 \setminus \{0\} : \langle \mathcal{J}_\lambda'(u), u \rangle = 0\}, \quad (1.5)$$

where \langle, \rangle is the duality between X_0 and its dual space. First we show that, the set \mathcal{N}_λ is a manifold for small λ .

Theorem 1.1. There exists $\lambda_0 > 0$ such that \mathcal{N}_λ is a C^1 manifold for all $\lambda \in (0, \lambda_0)$.

We show the following existence and multiplicity theorems for all $\lambda \in (0, \lambda_0)$ in the subcritical case. We remark that these results are sharp in the sense that for $\lambda > \lambda_0$, it is not clear if \mathcal{N}_λ is a manifold.

Theorem 1.2. Let $r \in (2p, p^*)$. Then, (P_λ) has at least two solutions for $\lambda \in (0, \lambda_0)$.

For the second theorem, we consider the following minimization problem

$$\Lambda = \inf \left\{ \|u\|_{X_0}^{2p} : u \in X_0, \int_{\Omega} g(x)|u|^{2p} dx = 1 \right\}. \quad (1.6)$$

Using direct methods of calculus of variations, we can show that $\Lambda > 0$ and is achieved by $u_\Lambda \geq 0$ in Ω .

Theorem 1.3. Let $r = 2p$ and let Λ be as in (1.6). Then

(i) (P_λ) has at least one solution for all $\lambda > 0$.

(ii) For $b < \frac{1}{\Lambda}$, there exists a $\lambda^0 > 0$ such that (P_λ) has at least two solutions for $\lambda \in (0, \lambda^0)$.

For our next result, we use the following version of Lemma 2 of [2].

Lemma 1.1. Let u be a non-negative solution of

$$M(\|u\|_{X_0}^p)(-\Delta)_p^s u = h_\lambda(x, u) \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \quad (1.7)$$

where $h_\lambda \in C(\Omega \times \mathbb{R})$ satisfies

$$h_\lambda(x, t) \leq \lambda C_0 |t|^q + C_1 |t|^r \text{ for all } x \in \Omega, t \in \mathbb{R} \quad (1.8)$$

and $C_0 \geq 0$, $C_1 > 0$, $\lambda > 0$, $0 < q < r$, $1 < r < p_s^* - 1$. Then there exists $C_* > 0$, independent of M , such that

$$\|u\|_{X_0}^p < \max\{M(\|u\|_{X_0}^p)^{\frac{q-r+2}{r-1}}, M(\|u\|_{X_0}^p)^{\frac{2}{r-1}}\}(\lambda C_0 C_*^{q+1} + C_1 C_*^{r+1})|\Omega|. \quad (1.9)$$

Proof. Let u be a non-negative solution of (1.7). Then $v = \frac{u}{M(\|u\|_{X_0}^p)^{\frac{1}{r-1}}}$ is a non-negative solution of

$$(-\Delta)_p^s v = g_\lambda(x, v) \text{ in } \Omega, \quad v = 0 \text{ in } \mathbb{R}^n \setminus \Omega.$$

By (1.8), and the fact that $M(s) \geq a$ for all s , we see that

$$\begin{aligned} |g_\lambda(x, s)| &= \frac{|h_\lambda(x, M(\|u\|_{X_0}^p)^{\frac{1}{r-1}} s)|}{M(\|u\|_{X_0}^p)^{\frac{p-1}{r-1}+1}} \\ &\leq M^{\frac{q-p+1}{r-1}-1} \lambda C_0 |s|^q + M^{\frac{r-p+1}{r-1}-1} C_1 |s|^r \\ &\leq a^{\frac{r-q+p-2}{r-1}} \lambda C_0 |s|^q + a^{\frac{p-2}{r-1}} C_1 |s|^r \leq \lambda C_2 |s|^q + C_3 |s|^r, \end{aligned}$$

for some positive constants C_2 and C_3 . Now for $r < 2p$, we have $1 + \frac{q}{p} > \frac{r}{p} + \frac{r}{p_s^*}$. So by Theorem 3.1 of [19], $\|v\|_\infty \leq C_*$, for some $C_* > 0$ (independent of M). Therefore

$$\|u\|_\infty \leq M(\|u\|_{X_0}^p)^{\frac{1}{r-1}} C_*.$$

Since u solves (1.7), multiplying (1.7) by u and integrating by parts, we get

$$\begin{aligned} \|u\|_{X_0}^p &= M(\|u\|_{X_0}^p)^{-1} \int_\Omega |h_\lambda(x, u) u| dx \\ &\leq M(\|u\|_{X_0}^p)^{-1} (\lambda C_0 \|u\|_\infty^{q+1} + C_1 \|u\|_\infty^{r+1}) |\Omega| \\ &\leq \max\{M(\|u\|_{X_0}^p)^{\frac{q-r+2}{r-1}}, M(\|u\|_{X_0}^p)^{\frac{2}{r-1}}\} (\lambda C_0 C_*^{q+1} + C_1 C_*^{r+1}) |\Omega|. \end{aligned}$$

□

Now for the case $r < 2p$, we define $L(\lambda) = (\lambda C_0 C_*^{q+1} + C_1 C_*^{r+1}) |\Omega|$ and

$$\hat{A} = \max_{k \in I} \left\{ M(k)^{\frac{q-r+2}{r-1}}, M(k)^{\frac{2}{r-1}} \right\}, \text{ where } I = \left(\frac{a(r-p)}{rb}, \frac{a(r-p)}{pb} \right). \quad (1.10)$$

Then we have the following theorem.

Theorem 1.4. *Let $r < 2p$ and let \hat{A} be defined in (1.10). Then*

(i) (P_λ) has at least one solution for each $\lambda > 0$.

(ii) For any $\theta > 0$, and $0 < b < \frac{a(r-p)}{r\widehat{AL}(\theta)}$, there exists $\hat{\lambda}_0 \in (0, \theta]$ such that (P_λ) has at least two solutions for each $0 < \lambda < \hat{\lambda}_0$.

Finally, in the case of critical nonlinearity, we have the following theorem.

Theorem 1.5. Let $r = p_s^*$ and $g(x) \equiv 1$. Then there exists a $\lambda_{00} > 0$ such that (P_λ) admits at least one solution for all $\lambda \in (0, \lambda_{00})$.

The paper is organized as follows: In section 2, we introduce Nehari manifold for (P_λ) and show that the associated Euler functional is bounded below on this manifold. Section 3 contains the existence and multiplicity results in the subcritical case. In section 4, we have the existence of nontrivial solution in the critical case.

2 Nehari manifold for (P_λ)

In this section we describe the nature of Nehari manifold corresponding to the problem (P_λ) . In the case $r \geq 2p$, the functional \mathcal{J}_λ is not bounded below on X_0 since $\widehat{M}(t) \sim t^{2p}$ as $t \rightarrow \infty$. We will show that it is bounded on some suitable subset of X_0 and on minimizing \mathcal{J}_λ on these subsets, we get the solutions for problem (P_λ) . From the definition of \mathcal{N}_λ , $u \in \mathcal{N}_\lambda$ if and only if

$$M(\|u\|_{X_0}^p) \int_Q |u(x) - u(y)|^p |x - y|^{-n-ps} dx dy - \lambda \int_\Omega f(x) |u|^q dx - \int_\Omega g(x) |u|^r dx = 0. \quad (2.1)$$

We note that \mathcal{N}_λ contains every non zero solution of (P_λ) . Now as we know that the Nehari manifold is closely related to the behavior of the functions $\phi_u : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined as $\phi_u(t) = \mathcal{J}_\lambda(tu)$. Such maps are called fiber maps and were introduced by Drabek and Pohozaev in [12]. For $u \in X_0$, we have

$$\phi_u(t) = \frac{1}{p} \widehat{M}(t^p \|u\|_{X_0}^p) - \frac{\lambda}{q} t^q \int_\Omega f(x) |u|^q dx - \frac{1}{r} t^r \int_\Omega g(x) |u|^r dx, \quad (2.2)$$

$$\phi'_u(1) = M(\|u\|_{X_0}^p) \|u\|_{X_0}^p - \lambda \int_\Omega f(x) |u|^q dx - \int_\Omega g(x) |u|^r dx, \quad (2.3)$$

$$\begin{aligned} \phi''_u(1) &= (p-1)M(\|u\|_{X_0}^p) \|u\|_{X_0}^p + M'(\|u\|_{X_0}^p) p \|u\|_{X_0}^{2p} \\ &\quad - \lambda(q-1) \int_\Omega f(x) |u|^q dx - (r-1) \int_\Omega g(x) |u|^r dx. \end{aligned} \quad (2.4)$$

Then it is easy to see that $u \in \mathcal{N}_\lambda$ if and only if $\phi'_u(1) = 0$. Thus it is natural to split \mathcal{N}_λ into three parts corresponding to local minima, local maxima and points of inflection. For this, we set

$$\begin{aligned} \mathcal{N}_\lambda^\pm &:= \{u \in \mathcal{N}_\lambda : \phi''_u(1) \gtrless 0\} = \left\{tu \in X_0 : \phi'_u(t) = 0, \phi''_u(t) \gtrless 0\right\}, \\ \mathcal{N}_\lambda^0 &:= \{u \in \mathcal{N}_\lambda : \phi''_u(1) = 0\} = \left\{tu \in X_0 : \phi'_u(t) = 0, \phi''_u(t) = 0\right\}. \end{aligned}$$

We define $H^\pm = \{u \in X_0 : \int_\Omega f(x)|u|^q dx \gtrless 0\}$, $G^\pm = \{u \in X_0 : \int_\Omega g(x)|u|^r dx \gtrless 0\}$, $H^0 = \{u \in X_0 : \int_\Omega f(x)|u|^q dx = 0\}$ and $G^0 = \{u \in X_0 : \int_\Omega g(x)|u|^r dx = 0\}$. Define $\psi_u : \mathbb{R}^+ \rightarrow \mathbb{R}$ as

$$\psi_u(t) = at^{p-q}\|u\|_{X_0}^p + bt^{2p-q}\|u\|_{X_0}^{2p} - t^{r-q} \int_\Omega g(x)|u|^r dx. \quad (2.5)$$

Then

$$\psi'_u(t) = a(p-q)t^{p-q-1}\|u\|_{X_0}^p + b(2p-q)t^{2p-q-1}\|u\|_{X_0}^{2p} - (r-q)t^{r-q-1} \int_\Omega g(x)|u|^r dx. \quad (2.6)$$

We also note that ϕ_{tu} and ψ_u satisfies

$$\phi''_{tu}(1) = t^{-q-1}\psi'_u(t). \quad (2.7)$$

Lemma 2.1. *If u is a minimizer of \mathcal{J}_λ on \mathcal{N}_λ such that $u \notin \mathcal{N}_\lambda^0$. Then u is a critical point for \mathcal{J}_λ .*

Proof. The details of the proof can be found in [12]. □

Define $\theta_\lambda := \inf\{\mathcal{J}_\lambda(u) | u \in \mathcal{N}_\lambda\}$ and $\theta_\lambda^\pm := \inf\{\mathcal{J}_\lambda(u) | u \in \mathcal{N}_\lambda^\pm\}$. Then

Lemma 2.2. *\mathcal{J}_λ is coercive and bounded below on \mathcal{N}_λ .*

Proof. For $u \in \mathcal{N}_\lambda$, using Hölder's inequality, we have

$$\begin{aligned} \mathcal{J}_\lambda(u) &= \left(\frac{1}{p} - \frac{1}{r}\right) a\|u\|_{X_0}^p + \left(\frac{1}{2p} - \frac{1}{r}\right) b\|u\|_{X_0}^{2p} - \left(\frac{1}{q} - \frac{1}{r}\right) \int_\Omega f(x)|u|^q dx, \\ &\geq \left(\frac{1}{p} - \frac{1}{r}\right) a\|u\|_{X_0}^p + \left(\frac{1}{2p} - \frac{1}{r}\right) b\|u\|_{X_0}^{2p} - \lambda \left(\frac{1}{q} - \frac{1}{r}\right) l^{\frac{r-q}{r}} \left(\int_\Omega |u|^r dx\right)^{q/r}, \\ &\geq \left(\frac{1}{p} - \frac{1}{r}\right) a\|u\|_{X_0}^p + \left(\frac{1}{2p} - \frac{1}{r}\right) b\|u\|_{X_0}^{2p} - \lambda \left(\frac{1}{q} - \frac{1}{r}\right) l^{\frac{r-q}{r}} S_r^{-q} \|u\|_{X_0}^q \end{aligned}$$

Thus \mathcal{J}_λ is coercive and bounded below in \mathcal{N}_λ for $r > 2p$. □

Lemma 2.3. *There exists $\lambda_0 > 0$ such that $\mathcal{N}_\lambda^0(\Omega) = \emptyset$, $\forall \lambda \in (0, \lambda_0)$*

Proof. We have following two cases.

Case 1: $u \in \mathcal{N}_\lambda(\Omega)$ and $\int_\Omega f(x)|u|^q dx = 0$.

From equation (2.3), we have, $a\|u\|_{X_0}^p + b\|u\|_{X_0}^{2p} - \int_\Omega g(x)|u|^r dx = 0$. Now,

$$\begin{aligned} pa\|u\|_{X_0}^p + 2pb\|u\|_{X_0}^{2p} - r \int_\Omega g(x)|u|^r dx &= pa\|u\|_{X_0}^p + 2pb\|u\|_{X_0}^{2p} - r(a\|u\|_{X_0}^p + b\|u\|_{X_0}^{2p}) \\ &= (p-r)a\|u\|_{X_0}^p + (2p-r)b\|u\|_{X_0}^{2p} < 0. \end{aligned}$$

which implies $u \notin \mathcal{N}_\lambda^0(\Omega)$

Case 2: $u \in \mathcal{N}_\lambda(\Omega)$ and $\int_\Omega f(x)|u|^q dx \neq 0$.

Suppose $u \in \mathcal{N}_\lambda^0(\Omega)$. Then from equations (2.3) and (2.4), we have

$$(p-q)a\|u\|_{X_0}^p + (2p-q)b\|u\|_{X_0}^{2p} = (r-q) \int_\Omega g(x)|u|^r dx, \quad (2.8)$$

$$(r-p)a\|u\|_{X_0}^p + (r-2p)b\|u\|_{X_0}^{2p} = (r-q)\lambda \int_\Omega f(x)|u|^q dx. \quad (2.9)$$

Define $E_\lambda : \mathcal{N}_\lambda(\Omega) \rightarrow \mathbb{R}$ as

$$E_\lambda(u) = \frac{(r-p)a\|u\|_{X_0}^p + (r-2p)b\|u\|_{X_0}^{2p}}{(r-q)} - \lambda \int_\Omega f(x)|u|^q dx,$$

then, from equation (2.9), $E_\lambda(u) = 0$, $\forall u \in \mathcal{N}_\lambda^0(\Omega)$. Also,

$$\begin{aligned} E_\lambda(u) &\geq \left(\frac{r-p}{r-q} \right) a\|u\|_{X_0}^p - \lambda \int_\Omega f(x)|u|^q dx \\ &\geq \left(\frac{r-p}{r-q} \right) a\|u\|_{X_0}^p - \lambda \|f\|_{L^{\frac{r}{r-q}}(\Omega)} \|u\|_{X_0}^q S_r^{-q}, \\ &\geq \|u\|_{X_0}^q \left[\left(\frac{r-p}{r-q} \right) a\|u\|_{X_0}^{(p-q)} - \lambda \|f\|_{L^{\frac{r}{r-q}}(\Omega)} S_r^{-q} \right], \end{aligned}$$

Now, from equation (2.8), we get

$$\|u\| \geq \left[\left(\frac{p-q}{r-q} \right) \frac{aS_r^r}{\|g\|_\infty} \right]^{\frac{1}{r-p}}. \quad (2.10)$$

Using equation (2.10), we get

$$E_\lambda(u) \geq \|u\|_{X_0}^q \left[(r-p) \left(\frac{a}{r-q} \right)^{\frac{r-q}{r-p}} \left(\frac{(p-q)S_r^r}{\|g\|_\infty} \right)^{\frac{p-q}{r-p}} - \lambda \|f\|_{L^{\frac{r}{r-q}}(\Omega)} S_r^{-q} \right].$$

This implies that there exists $\lambda_1 > 0$ such that for $\lambda \in (0, \lambda_1)$, $E_\lambda(u) > 0$, $\forall u \in \mathcal{N}_\lambda^0(\Omega)$, which is contradiction. Hence, $\mathcal{N}_\lambda^0(\Omega) = \emptyset$.

□

Proof of Theorem 1.1: Now the proof of Theorem 1.1 follows from the implicit function theorem and Lemma 2.3.

3 Existence and multiplicity results in the subcritical case

In this section, we prove the existence and multiplicity results for the case $r < p_s^*$. We need the following Lemmas.

Lemma 3.1. *Every Palaise-Smale sequence of \mathcal{J}_λ has a convergent subsequence. That is, if $\{u_k\} \subset X_0$ satisfies*

$$\mathcal{J}_\lambda(u_k) = c + o_k(1) \text{ and } \mathcal{J}'_\lambda(u_k) = o_k(1) \text{ in } X'_0, \quad (3.1)$$

then $\{u_k\}$ has a convergent subsequence.

Proof. Let $\{u_k\}$ be a sequence satisfying equation (3.1). Then it is easy to verify that $\{u_k\}$ is bounded in X_0 . So upto subsequence $u_k \rightharpoonup u_\lambda$ weakly in X_0 , $u_k \rightarrow u_\lambda$ strongly in $L^q(\Omega)$, $1 \leq q < p_s^*$ and $u_k(x) \rightarrow u_\lambda(x)$ a.e. in Ω . Now, by compactness of the imbedding $X_0 \hookrightarrow L^\alpha(\Omega)$ for all $\alpha < p_s^*$, we have

$$\int_\Omega f(x)|u_k|^q dx \rightarrow \int_\Omega f(x)|u_\lambda|^q dx \text{ as } k \rightarrow \infty.$$

Therefore, $\langle \mathcal{J}'_\lambda(u_k) - \mathcal{J}'_\lambda(u_\lambda), (u_k - u_\lambda) \rangle \rightarrow 0$ as $k \rightarrow \infty$. Now using the inequality $|a - b|^l \leq 2^{l-2}(|a|^{l-2}a - |b|^{l-2}b)(a - b)$ for $a, b \in \mathbb{R}^n$, $l \geq 2$, and $M(s) \geq a$ we get, $\|u_k - u_\lambda\|_{X_0} \rightarrow 0$ as $k \rightarrow \infty$. \square

Lemma 3.2. (i) *For every $u \in H^+ \cap G^+$, there is a unique $t_{\max} = t_{\max}(u) > 0$ and unique $t^+(u) < t_{\max} < t^-(u)$ such that $t^+u \in \mathcal{N}_\lambda^+$, $t^-u \in \mathcal{N}_\lambda$ and $\mathcal{J}_\lambda(t^+u) = \min_{0 \leq t \leq t^-} \mathcal{J}_\lambda(tu)$, $\mathcal{J}_\lambda(t^-u) = \max_{t \geq t_{\max}} \mathcal{J}_\lambda(tu)$.*

(ii) *For $u \in H^+ \cap G^-$, there exists a unique $t^* > 0$ such that $t^*u \in \mathcal{N}_\lambda^+$ and $\mathcal{J}_\lambda(t^*u) = \min_{t \geq 0} \mathcal{J}_\lambda(tu)$*

Proof. Let $u \in H^+ \cap G^+$. Then from equation (2.6), we note that $\psi_u(t) \rightarrow -\infty$ as $t \rightarrow \infty$. From equation (2.6), it is easy to see that $\lim_{t \rightarrow 0^+} \psi'_u(t) > 0$ and $\lim_{t \rightarrow \infty} \psi'_u(t) < 0$. So there exists a unique $t_{\max} = t_{\max}(u) > 0$ such that $\psi_u(t)$ is increasing on $(0, t_{\max})$, decreasing on (t_{\max}, ∞) and $\psi'_u(t_{\max}) = 0$.

$$\psi_u(t_{\max}) = t_{\max}^{-q} \left(a t_{\max}^p \|u\|_{X_0}^p + b t_{\max}^{2p} \|u\|_{X_0}^{2p} - t_{\max}^r \int_\Omega g(x)|u|^r \right)$$

where t_{\max} is the root of

$$a(p - q)t_{\max}^p \|u\|_{X_0}^p + b(2p - q)t_{\max}^{2p} \|u\|_{X_0}^{2p} - (r - q)t_{\max}^r \int_\Omega g(x)|u|^r = 0. \quad (3.2)$$

Now from equation (3.2), we get

$$t_{\max} \geq \frac{1}{\|u\|_{X_0}} \left[\frac{a(p - q)S_r^r}{(r - q)\|g\|_\infty} \right]^{\frac{1}{r-p}} := T_0 \quad (3.3)$$

Using inequality (3.3), we have

$$\begin{aligned}\psi_u(t_{\max}) &\geq \psi_u(T_0) \geq aT_0^{p-q}\|u\|_{X_0}^p - T_0^{r-q} \int_{\Omega} g(x)|u|^r dx \\ &\geq \|u\|_{X_0}^q (r-p) \left(\frac{a}{r-q}\right)^{\frac{r-q}{r-p}} \left(\frac{(p-q)S_r^r}{\|g\|_{\infty}}\right)^{\frac{p-q}{r-p}} > 0\end{aligned}$$

Hence if $\lambda < \lambda_2 = \left(\frac{(r-p)S_r^q}{\|f\|_{L^{\frac{r}{r-q}}(\Omega)}}\right) \left(\frac{a}{r-q}\right)^{\frac{r-q}{r-p}} \left(\frac{(p-q)S_r^r}{\|g\|_{\infty}}\right)^{\frac{p-q}{r-p}}$, then there exists unique $t^+ = t^+(u) < t_{\max}$ and $t^- = t^-(u) > t_{\max}$, such that $\psi_u(t^+) = \lambda \int_{\Omega} f(x)|u|^q = \psi_u(t^-)$. That is, $t^+u, t^-u \in \mathcal{N}_{\lambda}$. Also $\psi'_u(t^+) > 0$ and $\psi'_u(t^-) < 0$ implies $t^+u \in \mathcal{N}_{\lambda}^+$ and $t^-u \in \mathcal{N}_{\lambda}^-$. Since $\phi'_u(t) = t^q(\psi_u(t) - \lambda \int_{\Omega} f(x)|u|^q)$. Then $\phi'_u(t) < 0$ for all $t \in [0, t^+)$ and $\phi'_u(t) > 0$ for all $t \in (t^+, t^-)$. So $\mathcal{J}_{\lambda}(t^+u) = \min_{0 \leq t \leq t^+} \mathcal{J}_{\lambda}(tu)$. Also $\phi'_u(t) > 0$ for all $t \in [t^+, t^-)$, $\phi'_u(t^-) = 0$ and $\phi'_u(t) < 0$ for all $t \in (t^-, \infty)$ implies that $\mathcal{J}_{\lambda}(t^-u) = \max_{t \geq t_{\max}} \mathcal{J}_{\lambda}(tu)$.

(ii) Let $u \in H^+ \cap G^-$. Then from equation (2.6), we note that $\psi_u(t) \rightarrow \infty$ as $t \rightarrow \infty$. Also $\psi'_u(t) > 0$ for all $t > 0$. Hence for all $\lambda > 0$ there exists $t^* > 0$ such that $t^*u \in \mathcal{N}_{\lambda}^+$ and $\mathcal{J}_{\lambda}(t^*u) = \min_{t \geq 0} \mathcal{J}_{\lambda}(tu)$. □

Lemma 3.3. *There exists a constant $C_2 > 0$ such that*

$$\theta_{\lambda}^+ \leq -\frac{(p-q)(r-p)}{pqr} a C_2 < 0$$

Proof. Let $v_{\lambda} \in X_0$ such that $\int_{\Omega} f(x)|v_{\lambda}|^q dx > 0$. Then by Lemma 3.2, there exists $t_{\lambda}(v_{\lambda}) > 0$ such that $t_{\lambda}v_{\lambda} \in \mathcal{N}^+$. Since $t_{\lambda}v_{\lambda} \in \mathcal{N}_{\lambda}^+$, we have

$$\mathcal{J}_{\lambda}(t_{\lambda}v_{\lambda}) = \left(\frac{1}{p} - \frac{1}{q}\right) a \|t_{\lambda}v_{\lambda}\|_{X_0}^p + \left(\frac{1}{2p} - \frac{1}{q}\right) b \|t_{\lambda}v_{\lambda}\|_{X_0}^{2p} + \left(\frac{1}{q} - \frac{1}{r}\right) \int_{\Omega} g(x)|t_{\lambda}v_{\lambda}|^r dx.$$

and

$$\int_{\Omega} g(x)|t_{\lambda}v_{\lambda}|^r dx \leq \left(\frac{p-q}{r-q}\right) a \|t_{\lambda}v_{\lambda}\|_{X_0}^p + \left(\frac{2p-q}{r-q}\right) b \|t_{\lambda}v_{\lambda}\|_{X_0}^{2p}.$$

Therefore

$$\mathcal{J}_{\lambda}(t_{\lambda}v_{\lambda}) \leq -\frac{(p-q)(r-p)}{pqr} a \|t_{\lambda}v_{\lambda}\|_{X_0}^p \leq -\frac{(p-q)(r-p)}{pqr} a C_2,$$

where $C_2 = \|t_{\lambda}v_{\lambda}\|_{X_0}$. This implies $\theta_{\lambda}^+ \leq -\frac{(p-q)(r-p)}{pqr} a C_2$. □

Lemma 3.4. *For a given $u \in \mathcal{N}_{\lambda}$ and $\lambda \in (0, \lambda_0)$, there exists $\epsilon > 0$ and a differentiable function $\xi : \mathcal{B}(0, \epsilon) \subseteq X_0 \rightarrow \mathbb{R}$ such that $\xi(0) = 1$, the function $\xi(v)(u-v) \in \mathcal{N}_{\lambda}$ and*

$$\langle \xi'(0), v \rangle = \frac{pa\langle u, v \rangle + 2pb\|u\|_{X_0}^p \langle u, v \rangle - q\lambda \int_{\Omega} f(x)|u|^{q-2} u v dx - r \int_{\Omega} g(x)|u|^{r-2} u v dx}{(p-q)a\|u\|_{X_0}^p + (2p-q)b\|u\|_{X_0}^{2p} - (r-q) \int_{\Omega} g(x)|u|^r dx} \quad (3.4)$$

where $\langle u, v \rangle = \int_{\mathbb{R}^{2n}} |u(x) - u(y)|^{p-2} (u(x) - u(y))(v(x) - v(y)) |x - y|^{-n-ps} dx dy$ for all $v \in X_0$.

Proof. For fixed $u \in \mathcal{N}_\lambda$, define $\mathcal{F}_u : \mathbb{R} \times X_0 \rightarrow \mathbb{R}$ as follows

$$\mathcal{F}_u(t, w) = t^p a \|u - w\|_{X_0}^p + t^{2p} b \|u - w\|_{X_0}^{2p} - t^q \lambda \int_{\Omega} f(x) |u - w|^q dx - t^r \int_{\Omega} g(x) |u - w|^r dx$$

then $\mathcal{F}_u(1, 0) = 0$, $\frac{\partial}{\partial t} \mathcal{F}_u(1, 0) \neq 0$ as $\mathcal{N}_\lambda^0 = \emptyset$. So we can apply implicit function theorem to get a differentiable function $\xi : \mathcal{B}(0, \epsilon) \subseteq X_0 \rightarrow \mathbb{R}$ such that $\xi(0) = 1$ and equation (3.4) holds and $\mathcal{F}_u(\xi(w), w) = 0$, for all $w \in \mathcal{B}(0, \epsilon)$. This implies

$$a \|\xi(w)(u - w)\|_{X_0}^p + b \|\xi(w)(u - w)\|_{X_0}^{2p} - \lambda \int_{\Omega} f(x) |\xi(w)(u - w)|^q dx - \int_{\Omega} g(x) |\xi(w)(u - w)|^r dx = 0.$$

Hence $\xi(w)(u - w) \in \mathcal{N}_\lambda$. □

Proposition 3.1. *Let $\lambda_0 = \min\{\lambda_1, \lambda_2\}$, then for $\lambda \in (0, \lambda_0)$ there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}_\lambda$ such that*

$$\mathcal{J}_\lambda(u_k) = \theta_\lambda + o_k(1) \text{ and } \mathcal{J}'_\lambda(u_k) = o_k(1).$$

Proof. Using Lemma 2.2 and Ekeland variational principle [13], there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}_\lambda$ such that

$$\mathcal{J}_\lambda(u_k) < \theta_\lambda + \frac{1}{k} \tag{3.5}$$

$$\mathcal{J}_\lambda(u_k) < \mathcal{J}_\lambda(v) + \frac{1}{k} \|v - u_k\|_{X_0} \text{ for each } v \in \mathcal{N}_\lambda \tag{3.6}$$

Using equation (3.5), Lemma (3.3) and Hölder's inequality, we get, $u_k \not\equiv 0$. Next, we claim that $\|\mathcal{J}'_\lambda(u_k)\| \rightarrow 0$ as $k \rightarrow \infty$. Now, using the Lemma 3.4 we get the differentiable functions $\xi_k : \mathcal{B}(0, \epsilon_k) \rightarrow \mathbb{R}$ for some $\epsilon_k > 0$ such that $\xi_k(v)(u_k - v) \in \mathcal{N}_\lambda$, for all $v \in \mathcal{B}(0, \epsilon_k)$. For fixed k , choose $0 < \rho < \epsilon_k$. Let $u \in X_0$ with $u \not\equiv 0$ and let $v_\rho = \frac{\rho u}{\|u\|_{X_0}}$. We set $\eta_\rho = \xi_k(v_\rho)(u_k - v_\rho)$. Since $\eta_\rho \in \mathcal{N}_\lambda$, we get from equation (3.6)

$$\mathcal{J}_\lambda(\eta_\rho) - \mathcal{J}_\lambda(u_k) \geq -\frac{1}{k} \|\eta_\rho - u_k\|_{X_0}$$

Using mean value theorem, $\lim_{\rho \rightarrow 0} \frac{|\xi_k(v_\rho) - 1|}{\rho} \leq \|\xi_k'(0)\|_{X_0}$ and taking limit $\rho \rightarrow 0$, we get

$$\langle \mathcal{J}'_\lambda(u_k), \frac{u_k}{\|u_k\|_{X_0}} \rangle \leq \frac{C}{k} (1 + \|\xi_k'(0)\|_{X_0}) \tag{3.7}$$

for some constant $C > 0$, independent of ρ . So if we can show that $\|\xi_k'(0)\|_{X_0}$ is bounded then we are done. Now from Lemma 3.4 and Hölder's inequality, we get

$$\langle \xi_k'(0), v \rangle = \frac{K \|v\|_{X_0}}{(p - q)a \|u_k\|_{X_0}^p + (2p - q)b \|u_k\|_{X_0}^{2p} - (r - q) \int_{\Omega} g(x) |u_k|^r dx} \text{ for some } K > 0 \tag{3.8}$$

So to prove the claim we only need to prove that $(p - q)a\|u_k\|_{X_0}^p + (2p - q)b\|u_k\|_{X_0}^{2p} - (r - q) \int_{\Omega} g(x)|u_k|^r dx$ is bounded away from zero. Suppose not. Then there exists a subsequence u_k such that

$$(p - q)a\|u_k\|_{X_0}^p + (2p - q)b\|u_k\|_{X_0}^{2p} - (r - q) \int_{\Omega} g(x)|u_k|^r dx = o_k(1) \quad (3.9)$$

From equation (3.9), we get $E_{\lambda}(u_k) = o_k(1)$ and

$$\|u_k\|_{X_0} \geq \left[\left(\frac{p - q}{r - q} \right) \frac{aS_r^r}{\|g\|_{\infty}} \right]^{\frac{1}{r - p}} + o_k(1). \quad (3.10)$$

Now following the proof of Lemma 2.3, we get $E_{\lambda}(u_k) > 0$ for large k , which is a contradiction. Hence $\{u_k\}$ is a Palais-Smale sequence for \mathcal{J}_{λ} . □

Proof of Theorem 1.2 : Using proposition 3.1 and Lemma 3.1, there exist minimizing subsequences $\{u_k^{\pm}\} \in \mathcal{N}_{\lambda}^{\pm}$ and $u_{\lambda}^{\pm} \in X_0$ such that $u_k^{\pm} \rightarrow u_{\lambda}^{\pm}$ strongly in X_0 for $\lambda \in (0, \lambda_0)$. Therefore for $\lambda \in (0, \lambda_0)$, u_{λ}^{\pm} are weak solutions of problem (P_{λ}) . Hence $u_{\lambda}^{\pm} \in \mathcal{N}_{\lambda}$. Moreover $u_{\lambda}^{\pm} \in \mathcal{N}_{\lambda}^{\pm}$ and $\mathcal{J}_{\lambda}(u_{\lambda}^{\pm}) = \theta_{\lambda}^{\pm}$. As $\mathcal{J}_{\lambda}(u_{\lambda}^{\pm}) = \mathcal{J}_{\lambda}(|u_{\lambda}^{\pm}|)$ and $|u_{\lambda}^{\pm}| \in \mathcal{N}_{\lambda}^{\pm}$, by Lemma 2.1, u_{λ}^{\pm} are non-negative solutions of (P_{λ}) . Also, $\mathcal{N}_{\lambda}^{+} \cap \mathcal{N}_{\lambda}^{-} = \emptyset$, implies that u_{λ}^{+} and u_{λ}^{-} are distinct solutions.

Now to prove Theorem 1.3, we need following Lemmas.

Lemma 3.5. *Let $r = 2p$ and Λ be defined as in (1.6). Then*

(i) *if $b \geq \frac{1}{\Lambda}$, then $\mathcal{N}_{\lambda}^{+} = \mathcal{N}_{\lambda}$ for all $\lambda > 0$.*

(ii) *if $b < \frac{1}{\Lambda}$, then there exists a $\lambda^0 > 0$ such that $\mathcal{N}_{\lambda}^0 = \emptyset$, for all $\lambda \in (0, \lambda^0)$.*

Proof. (i) From equation (2.4), for any $u \in \mathcal{N}_{\lambda}$, we have

$$\begin{aligned} \phi_u''(1) &= a(p - q)\|u\|_{X_0}^p + (2p - q) \left(b\|u\|_{X_0}^{2p} - \int_{\Omega} g(x)|u|^{2p} dx \right) \\ &> a(p - q)\|u\|_{X_0}^p + \frac{(2p - q)(b\Lambda - 1)}{\Lambda} \|u\|_{X_0}^{2p} > 0. \end{aligned}$$

So $u \in \mathcal{N}_{\lambda}^{+}$, implies $\mathcal{N}_{\lambda}^{+} = \mathcal{N}_{\lambda}$.

(ii) Suppose $u \in \mathcal{N}_{\lambda}^0$. Then by equation (2.4)

$$(2p - q) \left(\int_{\Omega} g(x)|u|^{2p} dx - b\|u\|_{X_0}^{2p} \right) = a(p - q)\|u\|_{X_0}^p \quad (3.11)$$

$$\|u\|_{X_0} \geq \left(\frac{a(p - q)\Lambda}{(2p - q)(1 - b\Lambda)} \right)^{\frac{1}{p}}. \quad (3.12)$$

Also, from equations (2.3), (2.4), Hölder's inequality and Sobolev inequality we get ,

$$\|u\|_{X_0} \leq \left(\frac{\lambda(2p-q)l^{\frac{r}{r-q}}}{apS_r^q} \right)^{\frac{1}{p-q}}, \quad (3.13)$$

where S_r is the Sobolev constant for embedding of X_0 in $L^r(\Omega)$. Now, from equations (3.12) and (3.13), we get

$$\left(\frac{a(p-q)\Lambda}{(2p-q)(1-b\Lambda)} \right)^{\frac{1}{p}} \leq \|u\|_{X_0} \leq \left(\frac{\lambda(2p-q)l^{\frac{r}{r-q}}}{apS_r^q} \right)^{\frac{1}{p-q}}. \quad (3.14)$$

So equation (3.14) holds if $\lambda \geq \lambda^0 = \frac{paS_r^q}{(2p-q)l^{\frac{r}{r-q}}} \left(\frac{a\Lambda(p-q)}{(1-b\Lambda)(2p-q)} \right)^{\frac{p-q}{p}}$. Thus, for $\lambda < \lambda^0$, $\mathcal{N}_\lambda^0 = \emptyset$ \square

Lemma 3.6. *For $r = 2p$ and $b \geq \frac{1}{\Lambda}$, there exists a unique $0 < t^+ < t_{\max}$ such that $t^+u \in \mathcal{N}_\lambda$, whenever $u \in H^+$. Also*

$$\mathcal{J}_\lambda(t^+u) = \inf_{t \geq 0} \mathcal{J}_\lambda(tu).$$

Proof. Using equation (2.5) and (2.6) for $r = 2p$, we have

$$\psi_u(t) = at^{p-q}\|u\|_{X_0}^p + t^{2p-q} \left(b\|u\|_{X_0}^{2p} - \int_{\Omega} g(x)|u|^{2p} dx \right), \quad (3.15)$$

$$\psi'_u(t) = a(p-q)t^{p-q-1}\|u\|_{X_0}^p + (2p-q)t^{2p-q-1} \left(b\|u\|_{X_0}^{2p} - \int_{\Omega} g(x)|u|^{2p} dx \right). \quad (3.16)$$

Observe that $\psi'_u(t) > 0$ for $t > 0$. So for $u \in H^+$ there exists a unique $t^+(u) > 0$ such that $t^+u \in \mathcal{N}_\lambda = \mathcal{N}_\lambda^+$. Now using equation (2.7), we get $\frac{d}{dt}\mathcal{J}_\lambda(t^+u) = 0$, $\frac{d^2}{dt^2}\mathcal{J}_\lambda(t^+u) > 0$ for all $t > 0$. Hence $\mathcal{J}_\lambda(t^+u) = \inf_{t \geq 0} \mathcal{J}_\lambda(tu)$. \square

Lemma 3.7. *For $r = 2p$ and $b \leq \frac{1}{\Lambda}$, there exists a unique $t_{\max}(u) > 0$ such that*

(i) *For $u \in H^+$, there exists $\lambda^0 > 0$ and unique $t^+(u) < t_{\max}(u) < t^-(u)$ such that $t^\pm u \in \mathcal{N}_\lambda^\pm$ and $\mathcal{J}_\lambda(t^+u) = \inf_{0 \leq t \leq t_{\max}} \mathcal{J}_\lambda(tu)$, $\mathcal{J}_\lambda(t^-u) = \sup_{t \geq t_{\max}} \mathcal{J}_\lambda(tu)$ for all $\lambda \in (0, \lambda^0)$.*

(ii) *For $u \in H^-$, there exists unique $t^-(u) > t_{\max}$ such that $t^-u \in \mathcal{N}_\lambda^-$ and $\mathcal{J}_\lambda(t^-u) = \sup_{t \geq 0} \mathcal{J}_\lambda(tu)$*

Proof. (i) From the equation (3.16) and (3.15), we have $t_{\max} > 0$ such that

$$\psi_u(t_{\max}) \geq \frac{pa^{\frac{2p-q}{p}}}{2p-q} \left(\frac{(p-q)\Lambda}{(2p-q)(1-\Lambda b)} \right)^{\frac{p-q}{p}} \|u\|_{X_0}^q$$

Now if $\lambda < \lambda^0$, we have unique $t^+ < t_{\max} < t^-$ such that $\psi_u(t^\pm) = \lambda \int_{\Omega} f(x)|u|^q dx$ that is, $t^\pm u \in \mathcal{N}_\lambda$. Since $\psi'_u(t) > 0$ for $0 < t < t_{\max}$ and $\psi'_u(t) < 0$ for $t > t_{\max}$ we get $\psi'_u(t^+) > 0$ and

$\psi'_u(t^-) < 0$. So by the equation (2.7), $t^\pm u \in \mathcal{N}_\lambda^\pm$.

Again from equation (2.7), we observe $\frac{d}{dt}\mathcal{J}_\lambda(t^+u) = 0$, $\frac{d^2}{dt^2}\mathcal{J}_\lambda(t^+u) > 0$ for all $0 < t < t_{\max}$. Hence $\mathcal{J}_\lambda(t^+u) = \inf_{0 < t < t_{\max}} \mathcal{J}_\lambda(tu)$. Similarly, $\frac{d}{dt}\mathcal{J}_\lambda(t^-u) = 0$, $\frac{d^2}{dt^2}\mathcal{J}_\lambda(t^-u) < 0$ for all $t > t_{\max}$. Hence $\mathcal{J}_\lambda(t^-u) = \sup_{t > t_{\max}} \mathcal{J}_\lambda(tu)$.

(ii) For $u \in H^-$, $\psi'_u(t) < 0$ for $t > t_{\max}$ and $\psi_u(t) \rightarrow -\infty$ as $t \rightarrow \infty$. So we have a unique $t^-(u) > t_{\max}$ such that $\psi_u(t^-) = \lambda \int_\Omega f(x)|u|^q dx$. From equation (2.7), $t^-u \in \mathcal{N}_\lambda^-$ and $\mathcal{J}_\lambda(t^-u) = \sup_{t \geq 0} \mathcal{J}_\lambda(tu)$. \square

Lemma 3.8. Assume $r = 2p$, $b < \frac{1}{\Lambda}$. Then $\theta_\lambda^+ < 0$.

Proof. Let $w_\lambda \in X_0$ be such that $\int_\Omega f(x)|w_\lambda|^q dx > 0$. Then from Lemma 3.7, $\mathcal{N}_\lambda^+ \neq \emptyset$. Take $u \in \mathcal{N}_\lambda^+$, we have

$$a\|u\|_{X_0}^p + b\|u\|_{X_0}^{2p} - \lambda \int_\Omega f(x)|u|^q dx - \int_\Omega g(x)|u|^{2p} dx = 0 \quad (3.17)$$

$$pa\|u\|_{X_0}^p + 2pb\|u\|_{X_0}^{2p} - \lambda q \int_\Omega f(x)|u|^q dx - 2p \int_\Omega g(x)|u|^{2p} dx > 0 \quad (3.18)$$

Now multiplying equation (3.17) by $2p$ and subtracting from equation (3.18), we get

$$\lambda(2p - q) \int_\Omega f(x)|u|^q dx \geq pa\|u\|_{X_0}^p \quad (3.19)$$

Now using equation (3.17) and (3.19), we get

$$\begin{aligned} \mathcal{J}_\lambda(u) &= \frac{1}{p} \widehat{M}(\|u\|_{X_0}^p) - \frac{\lambda}{q} \int_\Omega f(x)|u|^q dx - \frac{1}{2p} \int_\Omega g(x)|u|^{2p} dx, \\ &= a \left(\frac{1}{p} - \frac{1}{2p} \right) \|u\|_{X_0}^p - \lambda \left(\frac{1}{q} - \frac{1}{2p} \right) \int_\Omega f(x)|u|^q dx \\ &= \frac{a}{2p} \|u\|_{X_0}^p - \lambda \frac{(2p - q)}{2pq} \int_\Omega f(x)|u|^q dx \\ &= \frac{1}{2p} a \|u\|_{X_0}^p - \frac{1}{2q} a \|u\|_{X_0}^p < -\frac{(p - q)}{2pq} a \|u\|_{X_0}^p < 0. \end{aligned}$$

Hence $\theta_\lambda^+ < 0$. \square

Proof of Theorem 1.3: Using Lemmas 3.5, 3.6, 3.7, 3.8, 3.1 and repeating the argument given in proof of Theorem 1.2, we get two non-negative solutions $u_\lambda^+ \in \mathcal{N}_\lambda^+$ and $u_\lambda^- \in \mathcal{N}_\lambda^-$.

Proof of Theorem 1.4:

Now to prove Theorem 1.4, we use truncation technique as used in [2] and [11] for Laplacian

operator. We consider the following truncated problem,

$$(P_{\lambda,k}) \begin{cases} M_k \left(\int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} dx dy \right) (-\Delta)_p^s u = \lambda f(x) |u|^{q-2} u + g(x) |u|^{r-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \quad (3.20)$$

where $k \in \left(\frac{a(r-p)}{rb}, \frac{a(r-p)}{pb} \right)$ and

$$M_k(t) = \begin{cases} M(t) & \text{if } 0 \leq t \leq k, \\ M(k) & \text{if } t > k, \end{cases} \quad (3.21)$$

is a truncation of $M(t)$.

The energy functional associated to the problem is given as

$$\mathcal{J}_{\lambda,k}(u) = \frac{1}{p} \widehat{M}_k(\|u\|_{X_0}^p) - \frac{\lambda}{q} \int_{\Omega} f(x) |u|^q dx - \frac{1}{r} \int_{\Omega} g(x) |u|^r dx \quad (3.22)$$

where \widehat{M}_k is the primitive of M_k . We define

$$\mathcal{N}_{\lambda,k} = \{u \in X_0 \setminus \{0\} : \langle \mathcal{J}'_{\lambda,k}(u), u \rangle = 0\}$$

Now fiber maps $\phi_{u,k} : \mathbb{R}^+ \rightarrow \mathbb{R}$ are defined as $\phi_{u,k}(t) = \mathcal{J}_{\lambda,k}(tu)$. For $u \in X_0$, we have

$$\phi_{u,k}(t) = \frac{1}{p} \widehat{M}_k(t^p \|u\|_{X_0}^p) - \frac{\lambda}{q} t^q \int_{\Omega} f(x) |u|^q dx - \frac{1}{r} t^r \int_{\Omega} g(x) |u|^r dx, \quad (3.23)$$

$$\phi'_{u,k}(1) = M_k(\|u\|_{X_0}^p) \|u\|_{X_0}^p - \lambda \int_{\Omega} f(x) |u|^q dx - \int_{\Omega} g(x) |u|^r dx, \quad (3.24)$$

$$\phi''_{u,k}(1) = (p-1) M_k(\|u\|_{X_0}^p) \|u\|_{X_0}^p + M'_k(\|u\|_{X_0}^p) p \|u\|_{X_0}^{2p} \quad (3.25)$$

$$- \lambda(q-1) \int_{\Omega} f(x) |u|^q dx - (r-1) \int_{\Omega} g(x) |u|^r dx. \quad (3.26)$$

Then it is easy to see that $u \in \mathcal{N}_{\lambda,k}$ if and only if $\phi'_{u,k}(1) = 0$. That is,

$$M_k(\|u\|_{X_0}^p) \|u\|_{X_0}^p - \lambda \int_{\Omega} f(x) |u|^q dx - \int_{\Omega} g(x) |u|^r dx = 0. \quad (3.27)$$

We split $\mathcal{N}_{\lambda,k}$ into three parts corresponding to local minima, local maxima and points of inflection. For this, we set

$$\begin{aligned} \mathcal{N}_{\lambda,k}^{\pm} &:= \{u \in \mathcal{N}_{\lambda,k} : \phi''_{u,k}(1) \gtrless 0\} = \left\{ tu \in X_0 : \phi'_{u,k}(t) = 0, \phi''_{u,k}(t) \gtrless 0 \right\}, \\ \mathcal{N}_{\lambda,k}^0 &:= \{u \in \mathcal{N}_{\lambda,k} : \phi''_{u,k}(1) = 0\} = \left\{ tu \in X_0 : \phi'_{u,k}(t) = 0, \phi''_{u,k}(t) = 0 \right\}. \end{aligned}$$

We need following results to prove theorem 1.4.

Lemma 3.9. *The energy functional $\mathcal{J}_{\lambda,k}$ is coercive and bounded below on $\mathcal{N}_{\lambda,k}$.*

Proof. Using equation (3.27) for $u \in \mathcal{N}_{\lambda,k}$, we get

$$\begin{aligned}\mathcal{J}_{\lambda,k}(u) &= \frac{1}{p} \widehat{M}_k(\|u\|_{X_0}^p) - \frac{\lambda}{q} \int_{\Omega} f(x)|u|^q dx - \frac{1}{r} \int_{\Omega} g(x)|u|^r dx, \\ &= \frac{1}{p} \widehat{M}_k(\|u\|_{X_0}^p) - \frac{1}{r} M_k(\|u\|_{X_0}^p) \|u\|_{X_0}^p - \lambda \left(\frac{1}{q} - \frac{1}{r} \right) \int_{\Omega} f(x)|u|^q dx, \\ &> \left(\frac{a}{p} - \frac{M(k)}{r} \right) \|u\|_{X_0}^p - \lambda \left(\frac{1}{q} - \frac{1}{r} \right) l^{\frac{r}{r-q}} S_r^{-q} \|u\|_{X_0}^q.\end{aligned}$$

Thus $\mathcal{J}_{\lambda,k}$ is coercive and bounded below as $\frac{a}{p} > \frac{M(k)}{r}$ for selected choice of k . \square

Lemma 3.10. *There exists a $\lambda^1 > 0$ such that $\mathcal{N}_{\lambda,k}^0 = \emptyset$ for all $\lambda \in (0, \lambda^1)$.*

Proof. Suppose not, then for $u \in \mathcal{N}_{\lambda,k}^0$ we have two cases

Case 1: $\|u\|_{X_0}^p \leq k$

From equations (3.24) and (3.25), we have

$$(p-q)a\|u\|_{X_0}^p + (2p-q)b\|u\|_{X_0}^{2p} - (r-q) \int_{\Omega} g(x)|u|^r dx = 0 \quad (3.28)$$

$$(r-p)a\|u\|_{X_0}^p - (2p-r)b\|u\|_{X_0}^{2p} - \lambda(r-q) \int_{\Omega} f(x)|u|^q dx = 0 \quad (3.29)$$

Case 2: $\|u\|_{X_0}^p > k$

$$(p-q)M(k)\|u\|_{X_0}^p - (r-q) \int_{\Omega} g(x)|u|^r dx = 0. \quad (3.30)$$

$$(r-p)M(k)\|u\|_{X_0}^p - \lambda(r-q) \int_{\Omega} f(x)|u|^q dx = 0. \quad (3.31)$$

Now from equations (3.28), (3.30) and Hölder's inequality, we get

$$(p-q)a\|u\|_{X_0}^p \leq (r-q)\|g\|_{\infty} S_r^{-r} \|u\|_{X_0}^r. \quad (3.32)$$

$$(p-q)M(k)\|u\|_{X_0}^p \leq (r-q)\|g\|_{\infty} S_r^{-r} \|u\|_{X_0}^r. \quad (3.33)$$

From (3.32) and (3.33), we get

$$C_1\|u\|_{X_0}^p \leq (r-q)\|g\|_{\infty} S_r^{-r} \|u\|_{X_0}^r. \quad (3.34)$$

where $C_1 = (p-q) \min\{a, M(k)\}$. Similarly, from (3.29) and (3.31) and Hölder's inequality, we get

$$[(r-p)a - (2p-r)bk]\|u\|_{X_0}^p \leq \lambda(r-q)l^{\frac{r}{r-q}} S_r^{-q} \|u\|_{X_0}^q. \quad (3.35)$$

$$(r-p)M(k)\|u\|_{X_0}^p \leq \lambda(r-q)l^{\frac{r}{r-q}} S_r^{-q} \|u\|_{X_0}^q. \quad (3.36)$$

From (3.35) and (3.36), we get

$$C_2 \|u\|_{X_0}^p \leq \lambda(r-q) l^{\frac{r}{r-q}} S_r^{-q} \|u\|_{X_0}^q. \quad (3.37)$$

where $C_2 = \min\{[(r-p)a - (2p-r)bk], (r-p)M(k)\}$. From (3.34) and (3.37), we get

$$\left(\frac{C_1 S_r^r}{(r-q)\|g\|_\infty} \right)^{\frac{1}{r-p}} \leq \|u\|_{X_0} \leq \left(\frac{\lambda(r-q) l^{\frac{r}{r-q}}}{C_2 S_r^q} \right)^{\frac{1}{p-q}}. \quad (3.38)$$

Hence (3.38) holds if $\lambda \geq \lambda^1 = \left(\frac{C_1 S_r^r}{(r-q)\|g\|_\infty} \right)^{\frac{p-q}{r-p}} \left(\frac{C_2 S_r^q}{(r-q) l^{\frac{r}{r-q}}} \right)$. Hence for $\lambda < \lambda^1$, $\mathcal{N}_{\lambda,k}^0 = \phi$. \square

Next, we define $\theta_{\lambda,k}^+ = \inf_{u \in \mathcal{N}_{\lambda,k}} \mathcal{J}_{\lambda,k}(u)$. Then we have following Lemma.

Lemma 3.11. $\theta_{\lambda,k}^+ < 0$, for all $\lambda \in (0, \lambda^1)$.

Proof. Let $u \in \mathcal{N}_{\lambda,k}^+$. Then we have two cases.

Case 1: $\|u\|_{X_0}^p \leq k$

From (3.35), we have

$$[(r-p)a - (2p-r)bk] \|u\|_{X_0}^p \leq \lambda(r-q) \int_{\Omega} f(x) |u|^q dx. \quad (3.39)$$

Now, using (3.39), we have

$$\begin{aligned} \mathcal{J}_{\lambda,k}(u) &= \frac{1}{p} \widehat{M}_k(\|u\|_{X_0}^p) - \frac{1}{r} M_k(\|u\|_{X_0}^p) \|u\|_{X_0}^p - \lambda \left(\frac{1}{q} - \frac{1}{r} \right) \int_{\Omega} f(x) |u|^q dx \\ &= \left(\frac{1}{p} - \frac{1}{r} \right) a \|u\|_{X_0}^p + \left(\frac{1}{2p} - \frac{1}{r} \right) b \|u\|_{X_0}^{2p} - \lambda \left(\frac{1}{q} - \frac{1}{r} \right) \int_{\Omega} f(x) |u|^q dx \\ &\leq -\frac{(p-q)(r-p)}{pqr} a \|u\|_{X_0}^p + \frac{(2p-q)(2p-r)}{2pqr} b \|u\|_{X_0}^{2p} \\ &\leq -\frac{k}{pqr} ((r-p)a - (2p-r)bk) < 0. \end{aligned}$$

Case 2: $\|u\|_{X_0}^p > k$

From (3.31), we have

$$0 < (r-p)M(k) \|u\|_{X_0}^p < \lambda(r-q) \int_{\Omega} f(x) |u|^q dx. \quad (3.40)$$

Also

$$\widehat{M}_k(t) = \int_0^t M_k(s) ds = \int_0^k M(s) ds + \int_k^t M_k(s) ds = \widehat{M}(k) + M(k)(t-k) \text{ for } t > k.$$

Using (3.40), we get

$$\begin{aligned}\mathcal{J}_{\lambda,k}(u) &= \frac{1}{p}\widehat{M}_k(\|u\|_{X_0}^p) - \frac{1}{r}M_k(\|u\|_{X_0}^p)\|u\|_{X_0}^p - \lambda\left(\frac{1}{q} - \frac{1}{r}\right)\int_{\Omega} f(x)|u|^q dx \\ &= \frac{1}{p}[\widehat{M}(k) - M(k)k] + \left(\frac{1}{p} - \frac{1}{r}\right)M(k)\|u\|_{X_0}^p - \lambda\left(\frac{1}{q} - \frac{1}{r}\right)\int_{\Omega} f(x)|u|^q dx \\ &< -\frac{1}{2p}bk^2 - \lambda\frac{(r-q)(p-q)}{pqr}\int_{\Omega} f(x)|u|^q dx < 0,\end{aligned}$$

Hence $\theta_{\lambda,k}^+ < 0$. □

Proof of Theorem 1.4(i): Repeating the same argument as in Theorem 1.2 and using Lemma 3.1 and the fact that $\mathcal{J}_{\lambda,M}$ is bounded below in X_0 , we get a Palais-Smale sequence which converges strongly.

Proof of Theorem 1.4(ii): Let $\theta > 0$ and take $\lambda < \hat{\lambda}_0 = \min\{\theta, \lambda^1\}$. Then by Lemma 3.11 and proposition 3.1, there exists two minimizing sequences satisfying (3.1). Using Lemma 3.1, up to subsequences, $u_k^{\pm} \rightarrow u_{\lambda}^{\pm}$ strongly in X_0 . Hence $u_{\lambda}^{\pm} \in \mathcal{N}_{\lambda,k}^{\pm}$ and $\mathcal{J}_{\lambda,k}(u_{\lambda}^{\pm}) = \theta_{\lambda}^{\pm}$. As $\mathcal{J}_{\lambda,k}(u_{\lambda}^{\pm}) = \mathcal{J}_{\lambda,k}(|u_{\lambda}^{\pm}|)$ and $|u_{\lambda}^{\pm}| \in \mathcal{N}_{\lambda,k}^{\pm}$, by Lemma 2.1, u_{λ}^{\pm} are non-negative solutions of $P_{\lambda,k}$. Also, $\mathcal{N}_{\lambda,k}^+ \cap \mathcal{N}_{\lambda,k}^- = \phi$, implies that u_{λ}^+ and u_{λ}^- are distinct solutions of problem $P_{\lambda,k}$. Now we claim that $\|u_{\lambda,k}^{\pm}\|^p \leq k$. If not then by Lemma 1.1,

$$\begin{aligned}\frac{a(r-p)}{brL(\theta)} &< \frac{k}{L(\theta)} < \frac{\|u_{\lambda,k}^{\pm}\|_{X_0}^p}{L(\lambda)} \\ &< \max\{M(k)^{\frac{q-r+2}{r-1}}, M(k)^{\frac{2}{r-1}}\} = \hat{A}\end{aligned}$$

which implies $b > \frac{a(r-p)}{p\hat{A}L(\theta)}$, a contradiction. Hence $u_{\lambda}^{\pm} \in \mathcal{N}_{\lambda}^{\pm}$ and u_{λ}^+ and u_{λ}^- are distinct solutions of problem P_{λ} .

4 Existence of solution for critical case

To show the existence result in the case of critical nonlinearity, we need the following Lemma.

Lemma 4.1. *Suppose $\{u_j\}$ be a sequence in X_0 such that $\mathcal{J}_{\lambda}(u_j) \rightarrow c < \left(\frac{p_s^*-2p}{2pp_s^*}\right) \frac{(m_0C)^{n/ps}}{S^{(n-ps)/ps}} - C\lambda^{\frac{p}{p-q}}$ and $\mathcal{J}'_{\lambda}(u_j) \rightarrow 0$, where C is positive constant depending on p and q , then there exists a strongly convergent subsequence.*

Proof. By equation (3.1), $\{u_j\}$ is bounded in X_0 and so upto subsequence $\{u_j\}$ converges weakly to u in X_0 , strongly in L^q for all $1 \leq q < p_s^*$ and point wise to u almost everywhere in Ω . Also there exists $h \in L^p(\Omega)$ such that $|u_j(x)| \leq h(x)$ a.e. in Ω . Also, $\{\|u_j\|_{X_0}\}$ as a real sequence converges to α (say). Since M is continuous, $M(\|u_j\|_{X_0}^p) \rightarrow M(\alpha^p)$. Now we claim that

$$\|u_j\|_{X_0}^p \rightarrow \|u\|_{X_0}^p \quad \text{as } j \rightarrow +\infty, \quad (4.1)$$

Once claim is proved, we can invoke Brezis-Leib Lemma [5] to prove that u_j converges to u strongly in X_0 . We know that $\{u_j\}$ is also bounded in $W_0^{s,p}(\Omega)$. So we may assume that there exists two positive measures μ and ν on \mathbb{R}^n such that

$$|(-\Delta)_p^s u_j|^p dx \xrightarrow{*} \mu \quad \text{and} \quad |u_j|^{p_s^*} \rightharpoonup \nu, \quad (4.2)$$

in the sense of measure. Moreover, from ([22]), we have a countable index set J , positive constants $\{\nu_j\}_{j \in J}$ and $\{\mu_j\}_{j \in J}$ such that

$$\nu = |u|^{p_s^*} dx + \sum_{i \in J} \nu_i \delta_{x_i}, \quad \text{and} \quad (4.3)$$

$$\mu \geq |(-\Delta)_p^s u|^p dx + \sum_{i \in J} \mu_i \delta_{x_i}, \quad \nu_i \leq S \mu_i^{p_s^*/p}, \quad (4.4)$$

where S is the best constant of the embedding $W_0^{s,p}(\Omega)$ into $L^{p_s^*}(\Omega)$. Our goal is to show that J is empty. Suppose not, then there exists $i \in J$. For this x_i , define $\phi_\delta^i(x) = \phi(\frac{x-x_i}{\delta})$, $x \in \mathbb{R}^n$ and $\phi \in C_0^\infty(\mathbb{R}^n, [0, 1])$ such that $\phi = 1$ in $B(0, 1)$ and $\phi = 0$ in $\mathbb{R}^n \setminus B(0, 2)$. Since $\{\phi_\delta^i u_j\}$ is bounded in X_0 , we have $\mathcal{J}'_{a,\lambda}(u_j)(\phi_\delta^i u_j) \rightarrow 0$ as $j \rightarrow +\infty$. That is,

$$\begin{aligned} & M(\|u_j\|_{X_0}^p) \int_{\mathbb{R}^{2n}} u_j(x) |u_j(x) - u_j(y)|^{p-2} (u_j(x) - u_j(y)) (\phi_\delta^i(x) - \phi_\delta^i(y)) |x - y|^{-n-ps} dx dy \\ &= -M(\|u_j\|_{X_0}^p) \int_{\mathbb{R}^{2n}} \phi_\delta^i(y) |u_j(x) - u_j(y)|^p |x - y|^{-n-ps} dx dy \\ &+ \lambda \int_{\Omega} f(x) |u_j(x)|^{q-2} u_j(x) \phi_\delta^i(x) dx + \int_{\Omega} |u_j(x)|^{p_s^*} \phi_\delta^i(x) dx + o_j(1), \end{aligned} \quad (4.5)$$

as $j \rightarrow \infty$. Now using Hölder's inequality and the fact that $\{u_j\}$ is bounded in X_0 , we get

$$\begin{aligned} & \left| \int_{\mathbb{R}^{2n}} u_j(x) |u_j(x) - u_j(y)|^{p-2} (u_j(x) - u_j(y)) (\phi_\delta^i(x) - \phi_\delta^i(y)) |x - y|^{-n-ps} dx dy \right| \\ & \leq C \left(\int_{\mathbb{R}^{2n}} |u_j(x)|^p |\phi_\delta^i(x) - \phi_\delta^i(y)|^p |x - y|^{-n-ps} dx dy \right)^{\frac{1}{p}}. \end{aligned} \quad (4.6)$$

Now we claim that

$$\lim_{\delta \rightarrow 0} \left[\lim_{j \rightarrow +\infty} \left(\int_{\mathbb{R}^{2n}} |u_j(x)|^p |\phi_\delta^i(x) - \phi_\delta^i(y)|^p |x - y|^{-n-ps} dx dy \right) \right] = 0. \quad (4.7)$$

Using the Lipschitz regularity of ϕ_δ^i , we have, for some $L \geq 0$,

$$\int_{\mathbb{R}^{2n}} |u_j(x)|^p |\phi_\delta^i(x) - \phi_\delta^i(y)|^p |x - y|^{-n-ps} dx dy$$

$$\leq \frac{1}{\theta} \int_{\mathbb{R}^{2n}} |u_j(x)|^p |\phi_\delta^i(x) - \phi_\delta^i(y)|^p |x - y|^{-n-ps} dx dy \quad (4.8)$$

$$\begin{aligned} &\leq \frac{L^p \delta^{-p}}{\theta} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \cap \{|x-y| \leq \delta\}} |u_j(x)|^p |x - y|^{p-n-ps} dx dy \\ &\quad + \frac{2^p}{\theta} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \cap \{|x-y| > \delta\}} |u_j(x)|^p |x - y|^{-n-ps} dx dy \\ &\leq C \frac{(L^p \delta^{-p} + 2^p)}{\theta} \int_{\mathbb{R}^n} |h(x)|^p dx dy < +\infty, \end{aligned} \quad (4.9)$$

with $C = C(n, s, \delta) > 0$. So, by dominated convergence theorem

$$\begin{aligned} \lim_{j \rightarrow +\infty} \int_{\mathbb{R}^{2n}} |u_j(x)|^p |\phi_\delta^i(x) - \phi_\delta^i(y)|^p |x - y|^{-n-ps} dx dy \\ = \int_{\mathbb{R}^{2n}} |u(x)|^p |\phi_\delta^i(x) - \phi_\delta^i(y)|^p |x - y|^{-n-ps} dx dy. \end{aligned} \quad (4.10)$$

Now, following the calculations as in equation (4.8), we get

$$\begin{aligned} \int_{U \times V} |u(x)|^p |\phi_\delta^i(x) - \phi_\delta^i(y)|^p |x - y|^{-n-ps} dx dy \\ \leq \frac{L^p}{\theta} \delta^{-p} \int_U \int_{V \cap \{|x-y| \leq \delta\}} |u(x)|^p |x - y|^{p-n-ps} dx dy \\ + \frac{2^p}{\theta} \int_U \int_{V \cap \{|x-y| > \delta\}} |u(x)|^p |x - y|^{-n-ps} dx dy, \end{aligned} \quad (4.11)$$

where U and V are two generic subsets of \mathbb{R}^n . Next, we claim that

$$\int_{\mathbb{R}^{2n}} |u(x)|^p |\phi_\delta^i(x) - \phi_\delta^i(y)|^p |x - y|^{-n-ps} dx dy \rightarrow 0, \text{ as } \delta \rightarrow 0.$$

When $U = V = \mathbb{R}^n \setminus B(x_i, \delta)$ claim follows. When $U \times V = B(x_i, \delta) \times \mathbb{R}^n$ and $U \times V = \mathbb{R}^n \times B(x_i, \delta)$, we can use Proposition 2.1 of [20] to prove the claim. Thus

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}^{2n}} |u(x)|^p |\phi_\delta^i(x) - \phi_\delta^i(y)|^p |x - y|^{-n-ps} dx dy = 0. \quad (4.12)$$

Hence

$$M(\|u_j\|_{X_0}^p) \int_{\mathbb{R}^{2n}} u_j(x) |u_j(x) - u_j(y)|^{p-2} (u_j(x) - u_j(y)) (\phi_\delta^i(x) - \phi_\delta^i(y)) |x - y|^{-n-ps} dx dy \rightarrow 0, \quad (4.13)$$

as $\delta \rightarrow 0$ and $j \rightarrow \infty$. Now, using Hölder's inequality

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \frac{u_j(x) - u_j(y)}{|x - y|^{n+ps}} \right|^p &\leq 2^{p-1} \left[|u_j(y)|^p \left| \int_{\mathbb{R}^n \setminus \Omega} \frac{1}{|x - y|^{n+ps}} \right|^p + \left| \int_{\Omega} \frac{u_j(x) - u_j(y)}{|x - y|^{n+ps}} dx \right|^p \right] \\ &\leq C_1 |u_j(y)|^p + C_2 \int_{\Omega} |u_j(x) - u_j(y)|^p |x - y|^{-n-ps} dx, \end{aligned} \quad (4.14)$$

where $C_1 = 2^{p-1} \left| \int_{\mathbb{R}^n \setminus \Omega} \frac{dx}{|x-y|^{n+ps}} \right|^p$ and $C_2 = \frac{2^{p-1}}{\theta}$. Now using equation (4.14) and (4.2), we get

$$\begin{aligned} \liminf_{j \rightarrow +\infty} \int_{\mathbb{R}^n} \phi_{\delta}^i(y) \int_{\Omega} |u_j(x) - u_j(y)|^p K(x - y) dx dy \\ \geq C_3 \frac{1}{c(n, s)} \liminf_{j \rightarrow +\infty} \int_{\mathbb{R}^n} \phi_{\delta}^i(y) c(n, s) \left| \int_{\mathbb{R}^n} \frac{u_j(x) - u_j(y)}{|x - y|^{n+ps}} dx \right|^p dy \\ - C_4 \liminf_{j \rightarrow +\infty} \int_{\mathbb{R}^n} \phi_{\delta}^i(y) |u_j(y)|^p dy \\ \geq C_3 \frac{1}{c(n, s)} \int_{\mathbb{R}^n} \phi_{\delta}^i(y) d\mu - C_4 \int_{B(x_i, \delta)} |u(y)|^p dy, \end{aligned} \quad (4.15)$$

where $C_3 = \frac{1}{C_2}$ and $C_4 = \frac{C_1}{C_2}$. Moreover,

$$\int_{B(x_i, \delta)} f(x) |u_j(x)|^{q-2} u_j(x) \phi_{\delta}^i(x) dx \rightarrow \int_{B(x_i, \delta)} f(x) |u(x)|^{q-2} u(x) \phi_{\delta}^i(x) dx, \quad (4.16)$$

as $j \rightarrow +\infty$. We also observe that the integral goes to 0 as $\delta \rightarrow 0$. So, using equations (4.13), (4.15), (4.16) and (4.2) in equation (4.5), we get

$$\begin{aligned} \int_{\Omega} \phi_{\delta}^i(x) d\nu + \lambda \int_{B(x_i, \delta)} f(x) |u(x)|^{q-2} u(x) \phi_{\delta}^i(x) dx \\ \geq M(\alpha^p) C \left(\int_{\Omega} \phi_{\delta}^i(y) d\mu - \int_{B(x_i, \delta)} |u(y)|^p dy \right) + o_{\delta}(1). \end{aligned}$$

Now, by taking $\delta \rightarrow 0$, we conclude that $\nu_i \geq M(\alpha^p) C \mu_i \geq m_0 C \mu_i$. Then by (4.4), we get

$$\nu_i \geq \frac{(m_0 C)^{n/ps}}{S^{(n-ps)/ps}}, \quad (4.17)$$

for any $i \in J$, where $C = \frac{C_3}{c(n, s)}$, independent of λ . We will prove that equation (4.17) is not possible.

From the assumption in the Lemma, we get

$$\begin{aligned} c &> \left(\frac{p_s^* - 2p}{2pp_s^*} \right) \int_{\Omega} |u_j|^{p_s^*} + \frac{a}{2p} \|u_j\|_{X_0}^p - \lambda \left(\frac{2p - q}{2pq} \right) \int_{\Omega} f(x) |u_j|^q dx \\ &> \left(\frac{p_s^* - 2p}{2pp_s^*} \right) \int_{\Omega} |u_j|^{p_s^*} + \frac{a}{2p} \|u_j\|_{X_0}^{rq} - \lambda \left(\frac{2p - q}{2pq} \right) l^{\frac{p-q}{p}} S^{\frac{1}{r}} (\|u_j\|_{X_0}^{rq})^{\frac{1}{r}} \end{aligned}$$

for some $r = \frac{p}{q} > 1$. Now consider $h(t) = \frac{a}{2p}t^r - \lambda \left(\frac{2p-q}{2pq} \right) l^{\frac{p-q}{p}} S^{\frac{1}{r}} t$, which has minimizer at $t_0 = \left[\frac{\lambda(2p-q)l^{\frac{p-q}{p}} S^{\frac{1}{r}}}{ar} \right]^{\frac{1}{r-1}}$. Hence

$$\begin{aligned} c &> \left(\frac{p_s^* - 2p}{2pp_s^*} \right) \int_{\Omega} |u_j|^{p_s^*} - \left(\frac{1}{r^{\frac{1}{r-1}}} - \frac{1}{r^{\frac{r}{r-1}}} \right) \frac{(\lambda(2p-q)l^{\frac{p-q}{p}} S^{\frac{1}{r}})^{\frac{r}{r-1}}}{2pqa^{\frac{1}{r-1}}} \\ &> \left(\frac{p_s^* - 2p}{2pp_s^*} \right) \int_{\Omega} \phi_{\delta}^i |u_j|^{p_s^*} - C\lambda^{\frac{p}{p-q}} \end{aligned}$$

As $j \rightarrow \infty$ and $\delta \rightarrow 0$, we have

$$c > \left(\frac{p_s^* - 2p}{2pp_s^*} \right) \frac{(m_0 C)^{n/ps}}{S^{(n-ps)/ps}} - C\lambda^{\frac{p}{p-q}} \quad (4.18)$$

where $C = \left(\frac{1}{r^{\frac{1}{r-1}}} - \frac{1}{r^{\frac{r}{r-1}}} \right) \frac{((2p-q)l^{\frac{p-q}{p}} S^{\frac{1}{r}})^{\frac{r}{r-1}}}{2pqa^{\frac{1}{r-1}}}$, which is a contradiction. Therefore $\nu_i = 0$, $\forall i \in J$. Hence J is empty. Which implies $u_j \rightarrow u$ in $L^{p_s^*}(\Omega)$. So, by equation (3.1) taking $\phi = u_j$ and using dominated convergence theorem,

$$\lim_{j \rightarrow +\infty} M(\|u_j\|_{X_0}^p) \|u_j\|_{X_0}^p = \lambda \int_{\Omega} f(x) |u(x)|^q dx + \int_{\Omega} |u(x)|^{p_s^*} dx. \quad (4.19)$$

Now, we take $\phi = u$ in equation (3.1) and recalling that $M(\|u_j\|_{X_0}^p) \rightarrow M(\alpha^p)$, we get

$$M(\alpha^p) \|u_j\|_{X_0}^p = \lambda \int_{\Omega} f(x) u(x) dx - \int_{\Omega} |u(x)|^{p_s^*} dx. \quad (4.20)$$

So, combining equations (4.19) and (4.20), we get

$$M(\|u_j\|_{X_0}^p) \|u_j\|_{X_0}^p \rightarrow M(\alpha^p) \|u\|_{X_0}^p, \quad \text{as } j \rightarrow +\infty.$$

So, using this result, we have

$$\begin{aligned} M(\|u_j\|_{X_0}^p) (\|u_j\|_{X_0}^p - \|u\|_{X_0}^p) &= M(\|u_j\|_{X_0}^p) \|u_j\|_{X_0}^p - M(\alpha^p) \|u\|_{X_0}^p \\ &\quad - M(\|u_j\|_{X_0}^p) \|u\|_{X_0}^p + M(\alpha^p) \|u\|_{X_0}^p, \end{aligned} \quad (4.21)$$

which leads to

$$M(\|u_j\|_{X_0}^p) (\|u_j\|_{X_0}^p - \|u\|_{X_0}^p) \rightarrow 0. \quad (4.22)$$

Also

$$m_0(\|u_j\|_{X_0}^p - \|u\|_{X_0}^p) \leq M(\|u_j\|_{X_0}^p) (\|u_j\|_{X_0}^p - \|u\|_{X_0}^p), \quad (4.23)$$

which implies $\|u_j\|_{X_0}^p \rightarrow \|u\|_{X_0}^p$ and the claim is proved. Hence $u_j \rightarrow u$ strongly in X_0 . \square

Proof of Theorem 1.5: Assume $\lambda_0^1 = \max\{\lambda : \theta_\lambda \leq \left(\frac{p_s^* - 2p}{2pp_s^*}\right) \frac{(m_0 C)^{n/ps}}{S^{(n-ps)/ps}} - C\lambda^{\frac{p}{p-q}}\}$. and $\lambda_{00} = \min\{\lambda_0, \lambda_0^1\}$. Now by Lemma 2.2 and Ekeland variational principle, we get a minimizing sequence $\{u_k\}$ satisfying

$$\mathcal{J}_\lambda(u_k) = \theta_\lambda + o(1) \text{ and } \mathcal{J}'_\lambda(u_k) = o(1). \quad (4.24)$$

then $\{u_k\}$ is bounded in X_0 . Hence, upto subsequence, $u_k \rightharpoonup u_\lambda$ in X_0 , $u_k \rightarrow u_\lambda$ in $L^q \forall q \in [1, p_s^*)$ and $u_k(x) \rightarrow u_\lambda(x)$ a.e. in Ω . Now by Lemma 4.1, $u_k \rightarrow u_\lambda$ in X_0 for $\lambda \in (0, \lambda_{00})$. So by Lemma 2.1, u_λ is a solution of problem P_λ for $r = p_s^*$. Hence $u_\lambda \in \mathcal{N}_\lambda$. Now using equation (2.7), we get $u_\lambda \in \mathcal{N}_\lambda^+$.

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